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Regularity of (0, 1, ..., $r-2$, r) and (0, 1, ..., $r-2$, r)* Interpolations on Some Sets of the Unit Circle

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The purpose of this paper is to show regularity of $(0, 1, ..., r-2, r)$ and $(0, 1, ..., r-2, r)^*$ interpolations on the sets obtained by projecting vertically the zeros of $(1-x^2)\, P_n^{(\alpha,\beta)}(x)$ $(-1)<\alpha,\ \beta\leqslant\frac{1}{2})$, $(1-x)\, P_n^{(\alpha,\beta)}(x)$ $(-1<\alpha\leqslant\frac{1}{2},\ -1<\beta\leqslant -\frac{1}{2})$ and $(1+x)\, P_n^{(\alpha,\beta)}(x)$ $(-1<\alpha\leqslant -\frac{1}{2},\ -1<\beta\leqslant\frac{1}{2})$ respectively onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the nth Jacobi polynomial.

1. INTRODUCTION

Let $0=m_0<m_1<\cdots<m_q$ be integers, $Z_n=\{z_1, z_2, ..., z_n\}$ be the set of knots on the unit circle, and π_n be the set of polynomials of degree at most n. $(m_0, m_1, ..., m_q)$ interpolation on Z_n can be stated as following: for arbitrary complex numbers $\{c_{m_j,k};\ j=0, 1, ..., q,\ k=1, 2, ..., n\}$, does there exist a polynomial $Q_n\in\pi_{(q+1)n-1}$, satisfying

$$Q_n^{(m_j)}(z_k)=c_{m_j,k}, \qquad j=0, 1, ..., q, \quad k=1, 2, ..., n? \tag{1.1}$$

If for any set of numbers $c_{m_j,k}$ there exists a unique polynomial $Q_n\in\pi_{(q+1)n-1}$ satisfying (1.1), then we say that $(m_0, m_1, ..., m_q)$ interpolation on Z_n is regular (otherwise, is singular).

In 1960, O. Ǩis initiated the type of problem for the special knots $Z_n=\{z_k=e^{2\pi ik/n};\ k=1, 2, ..., n\}$, the nth roots of unity. He showed that $(0, 1, ..., r-2, r)$ interpolation on Z_n is regular [1]. Later, Sharma [2], [3] extended the results to $(0, m)$ and $(0, m_1, m_2)$ cases. In [4] and [5], Sharma and his associates considered regular, explicit representation and convergence problem of $(m_0, m_1, ..., m_q)$ interpolation on Z_n. It should be noted that the set of knots is always the nth roots of unity when one considers $(m_0, m_1, ..., m_q)$ interpolation on the set of the unit circle.

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Let $P_n^{(\alpha,\beta)}(x)(\alpha > -1,\ \beta > -1)$ denote the nth Jacobi polynomial with the normalization

$$P_n^{(\alpha,\beta)}(1) = \Gamma(\alpha+n+1)/n!\,\Gamma(\alpha+1).$$

Then it is easily seen that the nth roots of unity can be obtained by projecting vertically the zeros of $(1-x^2)\ P_{n/2}^{(1/2,\,1/2)}(x)$ (n even) or $(1-x)$ $P_{(n-1)/2}^{(1/2,\,-1/2)}(x)$ (n odd) onto the unit circle. Now we ask: is it regular for $(0, 1, ..., r-2, r)$ interpolation on the sets generated by $(1-x^2)\ P_n^{(\alpha,\beta)}(x)$, $(1-x)\ P_n^{(\alpha,\beta)}(x)$ and $(1+x)\ P_n^{(\alpha,\beta)}(x)$ respectively in the same way? In the next section we shall answer this problem for $(0, 1, ..., r-2, r)$ and $(0, 1, ..., r-2, r)^*$ interpolations in the case that the parameters α and β of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ are subject to the conditions respectively

$$-1 < \alpha \leqslant \tfrac{1}{2}, \qquad -1 < \beta \leqslant \tfrac{1}{2}, \tag{1.2}$$

$$-1 < \alpha \leqslant \tfrac{1}{2}, \qquad -1 < \beta \leqslant -\tfrac{1}{2}, \tag{1.3}$$

$$-1 < \alpha \leqslant -\tfrac{1}{2}, \qquad -1 < \beta \leqslant \tfrac{1}{2}. \tag{1.4}$$

2. Regularity

Let $z_0 = 1$, $z_{2n+1} = -1$, and $Z_{2n} = \{z_1, z_2, ..., z_{2n}\}$ satisfy

$$z_k = \cos\theta_k + i\sin\theta_k, \qquad z_{n+k} = \cos\theta_k - i\sin\theta_k, \qquad k = 1, 2, ..., n, \tag{2.1}$$

where $\{\cos\theta_k;\ k = 1, 2, ..., n\}$ are the zeros of $P_n^{(\alpha,\beta)}(x)$, with $1 > \cos\theta_1 > \cdots > \cos\theta_n > -1$, and $W(z) = \prod_{k=1}^{2n} (z - z_k)$ and $R(z) = (z^2-1)\ W(z)$. Since the coefficient of the first term of $P_n^{(\alpha,\beta)}(x)$ is [6]

$$\frac{1}{2^n n!}\ \frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)},$$

we know that

$$W(z) = K_n P_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right) z^n,$$

where

$$K_n = 2^{2n} n!\ \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}$$

Using the following relation [7]

$$(1-x^2)\frac{d^2}{dx^2}[P_n^{(\alpha,\beta)}(x)]+[\beta-\alpha-(\alpha+\beta+2)x]\frac{d}{dx}[P_n^{(\alpha,\beta)}(x)]$$
$$+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0,$$

we obtain

$$W'(z_k)=-\tfrac{1}{2}K_n(1-z_k^2)z_k^{n-2}P_n^{(\alpha,\beta)\prime}(\cos\theta_k),$$
$$k=1,2,...,n, \tag{2.2}$$
$$W'(z_{n+k})=-\tfrac{1}{2}K_n(1-z_{n+k}^2)z_{n+k}^{n-2}P_n^{(\alpha,\beta)\prime}(\cos\theta_k),$$
$$k=1,2,...,n, \tag{2.3}$$
$$W''(z_k)=[(2n-a-\beta-2)(1-z_k^2)+2(\alpha-\beta)z_k$$
$$+2(\alpha+\beta+1)](1-z_k^2)^{-1}z_k^{-1}W'(z_k),$$
$$k=1,2,...,2n. \tag{2.4}$$

From Leibniz' formula and (2.4) we have

$$[R(z)^{r-1}]^{(r-1)}|_{z=z_k}=(r-1)!\,(z_k^2-1)^{r-1}[W'(z_k)]^{r-1},$$
$$k=1,2,...,2n, \tag{2.5}$$
$$[R(z)^{r-1}]^{(r-1)}|_{z=z_k}=2^{r-1}(r-1)!\,z_k^{r-1}W(z_k)^{r-1},$$
$$k=0,2n+1, \tag{2.6}$$
$$[R(z)^{r-1}]^{(r)}|_{z=z_k}=(r-1)\,r!\,[(n-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+1)(z_k^2-1)$$
$$-(\alpha-\beta)z_k-(\alpha+\beta-1)]$$
$$\times z_k^{-1}(z_k^2-1)^{r-2}[W'(z_k)]^{r-1},$$
$$k=1,2,...,2n, \tag{2.7}$$
$$[R(z)^{r-1}]^{(r)}|_{z=z_k}=2^{r-2}(2n+1)(r-1)\,r!\,z_k^rW(z_k)^{r-1},$$
$$k=0,2n+1 \tag{2.8}$$

The main result in this paper is the following

THEOREM 1. (i) *If* α *and* β *satisfy* (1.2) *then* $(0, 1, ..., r-2, r)$ *interpolation on* $Z_{2n}\cup\{z_0, z_{2n+1}\}$ *is regular;*

(ii) *If* α *and* β *satisfy* (1.3) *then* $(0, 1, ..., r-2, r)$ *interpolation on* $Z_{2n}\cup\{z_0\}$ *is regular;*

(iii) *If* α *and* β *satisfy* (1.4) *then* $(0, 1, ..., r-2, r)$ *interpolation on* $Z_{2n}\cup\{z_{2n+1}\}$ *is regular.*

Proof. We only prove (i). The proof of (ii) and (iii) is similar. Obviously, it is sufficient to show the polynomial $Q_n \in \pi_{(2n+2)r-1}$ satisfying

$$Q_n^{(j)}(z_k)=0, \qquad j=0,\ 1,\ ...,\ r-2,\ r, \quad k=0,\ 1,\ ...,\ 2n+1, \tag{2.9}$$

vanishes. We know that $Q_n(z)=R(z)^{r-1}\,q(z)$ by (2.9), where $q(z)\in\pi_{2n+1}$, by (2.5)–(2.8) and $Q_n^{(r)}(z_k)=0$, $k=0,\ 1,\ ...,\ 2n+1$, we obtain

$$\begin{cases} z_k(1-z_k^2)\,q'(z_k)+(r-1)[(n-\frac{1}{2}\alpha-\frac{1}{2}\beta+1)(1-z_k^2) \\ \qquad +(\alpha-\beta)\,z_k+(\alpha+\beta-1)]\,q(z_k)=0, \\ \qquad k=1,\ 2,\ ...,\ 2n, \\ 2q'(1)+(2n+1)(r-1)\,q(1)=0, \\ 2q'(-1)-(2n+1)(r-1)\,q(-1)=0. \end{cases} \tag{2.10}$$

Therefore we can suppose

$$\begin{aligned} &z(1-z^2)\,q'(z)+(r-1) \\ &\qquad \times[(n-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+1)(1-z^2)+(\alpha-\beta)z+(\alpha+\beta-1)]\,q(z) \\ &=(a_0+a_1z+a_2z^2+a_3z^3)\,W(z), \end{aligned} \tag{2.11}$$

where a_0, a_1, a_2 and a_3 are constants. Substituting $z=1$ or -1 in (2.11) we have

$$\begin{cases} (r-1)(2\alpha-1)\,q(1)=(a_0+a_1+a_2+a_3)\,W(1), \\ (r-1)(2\beta-1)\,q(-1)=(a_0-a_1+a_2-a_3)\,W(-1). \end{cases} \tag{2.12}$$

Differentiating (2.11) once, then putting $z=1$ or -1, from (2.10) we obtain

$$\begin{cases} \frac{1}{2}(r-1)(2\alpha-1)[2-(2n+1)(r-1)]\,q(1) \\ \qquad =[na_0+(n+1)\,a_1+(n+2)\,a_2+(n+3)\,a_3]\,W(1), \\ -\frac{1}{2}(r-1)(2\beta-1)[2-(2n+1)(r-1)]\,q(-1) \\ \qquad =[-na_0+(n+1)\,a_1-(n+2)\,a_2+(n+3)\,a_3]\,W(-1), \end{cases} \tag{2.13}$$

(2.12) and (2.13) imply

$$\begin{cases} a_2=-\dfrac{\delta-1}{\delta+1}\,a_0, \\ a_3=-\dfrac{\delta}{\delta+2}\,a_1, \end{cases}$$

where $\delta = n + \frac{1}{2}(2n+1)(r-1)$. Hence we can write (2.11) as

$$z(1-z^2)\,q'(z) + (r-1)\left[\left(n - \frac{1}{2}\alpha - \frac{1}{2}\beta + 1\right)(1-z^2) + (\alpha-\beta)z + (\alpha+\beta-1)\right] q(z)$$
$$= a_0\left(1 - \frac{\delta-1}{\delta+1}z^2\right) W(z) + a_1\left(1 - \frac{\delta}{\delta+2}z^2\right) z W(z) \qquad (2.14)$$

Solving differential equations (2.14) we get

$$f(z)\,q(z) = a_0\,g(z) + a_1\,h(z), \qquad (2.15)$$

where

$$f(z) = z^{(r-1)(n+1/2\alpha+1/2\beta)}\,(1+z)^{-1/2(r-1)(2\beta-1)}\,(1-z)^{-1/2(r-1)(2\alpha-1)},$$

$$g(z) = \int_0^z t^{(r-1)(n+1/2\alpha+1/2\beta)-1}\,(1+t)^{-1/2(r-1)(2\beta-1)-1} \times (1-t)^{-1/2(r-1)(2\alpha-1)-1}\left(1 - \frac{\delta-1}{\delta+1}t^2\right) W(t)\,dt.$$

$$h(z) = \int_0^z t^{(r-1)(n+1/2\alpha+1/2\beta)}(1+t)^{-1/2(r-1)(2\beta-1)-1} \times (1-t)^{-1/2(r-1)(2\alpha-1)-1}\left(1 - \frac{\delta}{\delta+2}t^2\right) W(t)\,dt.$$

Firstly we suppose that $-1 < \alpha < \frac{1}{2}$ and $-1 < \beta < \frac{1}{2}$. Then

$$f(1) = f(-1) = 0,$$

therefore

$$\begin{cases} a_0\,g(1) + a_1\,h(1) = 0, \\ a_0\,g(-1) + a_1\,h(-1) = 0. \end{cases} \qquad (2.16)$$

Since $W(t)$ has no real zero and $W(0) = 1$, so $W(t) > 0$ if t is a real number. This yield.

$$g(1) > 0, \qquad h(1) > 0, \qquad (2.17)$$

$$(-1)^{(r-1)(n+1/2\alpha+1/2\beta)}\,g(-1) < 0, \qquad (-1)^{(r-1(n+1/2\alpha+1/2\beta)}\,h(-1) > 0. \qquad (2.18)$$

It follows from (2.16)–(2.18)

$$a_0 = a_1 = 0.$$

If $\alpha = \frac{1}{2}$, we have

$$f(1) < +\infty, \qquad g(1) = +\infty, \qquad h(1) = +\infty,$$

hence we can also derive $a_0 = a_1 = 0$ from (2.15). Similar result can be obtained in the case $\beta = \frac{1}{2}$. Up to now we have shown that $q(z) \equiv 0$. Hence $Q_n(z) \equiv 0$.

This completes the Proof of Theorem 1.

Now we consider the problem of $(0, 1, ..., r-1, r)^*$ interpolation defined analogously in [8] on $Z_{2n} \cup \{z_0, z_{2n+1}\}$, that is to decide whether or not there exists a unique polynomial $Q_n \in \pi_{(2n+2)r-3}$ satisfying

$$Q_n^{(j)}(z_k) = c_{jk}; \qquad j = 0, 1, ..., r-2, \quad k = 0, 1, ..., 2n+1,$$
$$Q_n^{(r)}(z_k) = c_{rk}; \qquad k = 1, 2, ..., 2n,$$

where $\{c_{jk}\}$ is an arbitrary set of numbers. By the similar method to the proof of theorem 1 we can prove

THEOREM 2. *If α and β satisfy* (1.2), *then* $(0, 1, ..., r-2, r)^*$ *interpolation on $Z_{2n} \cup \{z_0, z_{2n+1}\}$ is regular.*

Remark. The condition (1.2)–(1.4) may be removed in theorem 1 and 2. Unfortunately, the technique in the present paper does not seem to answer this question. Moreover, in another paper we have obtained convergence of $(0, 1, 3)^*$ interpolation on $Z_{2n} \cup \{z_0, z_{2n+1}\}$ in the case $\alpha = \beta = 0$.

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